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Spectral Bounds for $\|A^{-1}\|_\infty$

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It is shown that for positive definite Hermitian matrices A such that A^n has no more than kn nonzero entries in each row, the following upper bound holds: $\|A^{-1}\|_\infty \leqslant 33\sqrt{k}\, \|A^{-1}\|_2^{5/4}\, \|A\|_2^{1/4}$. The method of proof involves a Chebyshev expansion of A^{-1}.

1. For a square matrix A, $\|A\|_\infty$ denotes the norm of A considered as an operator on $l_\infty(I)$, where I is the index set of the rows of A. The formula $\|A\|_\infty = \sup_i \sum_j |A(i, j)|$ is well known. Similarly, $\|A\|_2$ denotes the norm of A as an operator on $l_2(I)$ and $\|A\|_2 = \sup\{\sqrt{|\lambda|}: \lambda \in \sigma(A^*A)\}$, where A^* is the conjugate transpose of A and $\sigma(B)$ denotes the spectrum of the operator B. In general, if A is an $n \times n$ matrix, we have $\|A\|_\infty \leqslant \sqrt{n}\, \|A\|_2$, or, if each row of A has no more than k nonzero entries, we have $\|A\|_\infty \leqslant \sqrt{k}\, \|A\|_2$ (cf. Lemma 2.3). If $A = A^*$, then $\|A\|_2 \leqslant \|A\|_\infty$ since $\sigma(A) \subseteq \{z: |z| \leqslant \|A\|\}$ for any operator norm. Thus, for self-adjoint- or even normal-matrices which are banded we see that the l_∞ and l_2 operator norms are equivalent with the constants of equivalence depending on only the bandwidth. In [1], we observed that an infinite matrix A which is banded and invertible in some l_p operator norm $(1 \leqslant p \leqslant \infty)$ is necessarily invertible in all l_p norms. This does not imply that these norms are uniformly equivalent in the sense that there exist constants $m = m(p, q)$, $M = M(p, q)$ independent of A^{-1} such that

$$m\, \|A^{-1}\|_q \leqslant \|A^{-1}\|_p \leqslant M\, \|A^{-1}\|_q.$$

Nor does the result of [1] or its sharpening in [2] imply that a sequence

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of invertible matrices all of the same bandwidth and uniformly bounded in some l_p operator norm must have inverses whose norms are asymptotically equivalent in all l_p norms. For example, if $\|A_N\|_2 = 1$ for all N and each A_N is tridiagonal, then we do not know if

$$0 < \liminf_N \frac{\|A_N^{-1}\|_2}{\|A_N^{-1}\|_\infty} \leqslant \limsup_N \frac{\|A_N^{-1}\|_2}{\|A_N^{-1}\|_\infty} < \infty.$$

Such matrices arise naturally in the numerical solution of boundary value problems by finite differences and finite element method. An understanding of how their inverse—or diagonal scalings of them—bound each other could be useful in establishing and unifying results on convergence in various norms.

In this paper we restrict attention to Hermitian matrices which are positive definite and banded but make no restriction on the size—finite or infinite or bi-infinite. The bandedness assumption can also be replaced by the assumption that each row of A^n has no more than nk nonzero entries for some fixed k. We call a matrix A *m-banded* if $A(i, j) = 0$ for $|i-j| > m/2$. So, tridiagonal matrices are 2-banded. Since our matrices are Hermitian there is no problem in identifying the diagonal in the bi-infinite versions. Our main result is Proposition 2.4, which asserts that for an m-banded invertible Hermitian A,

$$\|A^{-1}\|_\infty \leqslant 33\sqrt{m+1}\, \|A^{-1}\|_2^{5/4}\, \|A\|_2^{1/4}.$$

We conjecture that the 5/4 could be lowered to 1, perhaps at the expense of changing the remaining quantities. While the results are of an operator theoretic nature, the proofs are based on a notion of classical approximation theory—the Chebyshev series expansion of an analytic function. In addition, for us this whole subject of band matrices was motivated by problems in approximation theory.

2. We start with two technical lemmas.

LEMMA 2.1. *Let $0 < a < b$ and define*

$$q := \frac{\sqrt{b}-\sqrt{a}}{\sqrt{b}+\sqrt{a}} \tag{2.1}$$

and

$$c_n := \frac{2(-q)^n}{\sqrt{ab}} \qquad \textit{for} \quad n = 0, 1, 2, \ldots. \tag{2.2}$$

Then, for $a \leqslant x \leqslant b$ we have the following expansion

$$\frac{1}{x} = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n T_n(z)$$

where $z = (2/(b-a))(x-(a+b)/2)$ and the T_n's are the Chebyshev polynomials, $T_0 = 1$, $T_1(x) = x$, $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$.

Proof. $1/x = (2/(b-a)) \cdot (1/(z+c))$, where $c = (b+a)/(b-a)$. The Chebyshev coefficients

$$c_n = \frac{4}{(b-a)\pi} \int_0^{\pi} \frac{\cos n\theta}{\cos\theta + c}\, d\theta$$

can be computed exactly and give the above numbers (cf. [3, p. 366]).

LEMMA 2.2. *Let a, b, q be as above, then*

$$\sum_{n=1}^{\infty} q^n \sqrt{n} \leqslant 2^{5/2} \Gamma(3/2)(b/a)^{3/4}.$$

Proof. With $\alpha := -\ln q$ it's clear that $\sum_{n=1}^{\infty} q^n \sqrt{n}$ is not far from $\int_0^\infty x^{1/2} e^{-\alpha x}\, dx$. In fact

$$\sum_{n=1}^{\infty} q^n \sqrt{n} \leqslant 2 \int_0^\infty x^{1/2} e^{-\alpha x}\, dx$$

can be proven by breaking the series into two parts: one corresponding to $1 \leqslant x \leqslant 1/2\alpha$, where $x^{1/2} e^{-\alpha x}$ is increasing and one corresponding to $1/2\alpha \leqslant x \leqslant \infty$, where $x^{1/2} e^{-\alpha x}$ is decreasing. The change of variables $u = \alpha x$ gives

$$\sum_{n=1}^{\infty} q^n \sqrt{n} \leqslant 2 \int_0^\infty x^{1/2} e^{-\alpha x}\, dx = 2(\alpha)^{-3/2} \int_0^\infty u^{1/2} e^{-u}\, du$$

$$= \frac{2}{(\ln(1/q))^{3/2}} \Gamma(3/2).$$

Now,

$$q = \frac{\sqrt{b} - \sqrt{a}}{\sqrt{b} + \sqrt{a}} = \frac{1 - \sqrt{a/b}}{1 + \sqrt{a/b}} \leqslant \frac{1}{1 + \sqrt{a/b}},$$

so $1/q \geqslant 1+\sqrt{a/b}$ and

$$\frac{1}{\ln(1/q)} \leqslant \frac{1}{\ln(1+\sqrt{a/b})} = \frac{1}{\sum_{n=0}^{\infty}(-1)^n(\sqrt{a/b})^{n+1}\cdot(1/(n+1))}$$
$$\leqslant \frac{1}{\sqrt{a/b}(1-\frac{1}{2}\sqrt{a/b})}$$

Since $a \leqslant b$, $1/(1-\frac{1}{2}\sqrt{a/b}) \leqslant 2$. Thus, we have

$$\left\{\frac{1}{\ln(1/q)}\right\}^{3/2} \leqslant (2)^{3/2}\cdot(b/a)^{3/4}$$

which gives the desired result. ∎

For completeness we include the following result.

LEMMA 2.3. *Let B be r-banded. Then, B^k is rk-banded and* $\|B\|_\infty \leqslant \sqrt{r+1}\,\|B\|_2$.

Proof. The first statement is easy to check. Now, $\|B\|_\infty = \sup_i \sup\{\|Bx\|_\infty : \|x\|_\infty = 1$ and $x(j)=0$ for $|i-j|>r/2\} \leqslant \sup_i \sup\{\|Bx\|_2 : \|x\|_\infty = 1,\ x(j)=0$ for $|i-j|>r/2\} \leqslant \sqrt{r+1}\,\sup_i \sup\{\|Bx\|_2 : \|x\|_2 = 1,$ $x(j)=0$ for $|i-j|>r/2\} \leqslant \sqrt{r+1}\,\|B\|_2$. ∎

The main result now follows easily from elementary spectral theory.

PROPOSITION 2.4. *Let A be m-banded, Hermitian and positive definite. Then*

$$\|A^{-1}\|_\infty \leqslant 33\sqrt{m}\,\|A\|_2^{1/4}\,\|A^{-1}\|_2^{5/4}.$$

Proof. Let $a := \inf\{\langle Ax, x\rangle : \|x\|_2 = 1\}$ and $b := \sup\{\langle Ax, x\rangle : \|x\|_2 = 1\}$, so $\|A^{-1}\|_2 = a^{-1}$ and $\|A\|_2 = b$. If $b=a$, then A is diagonal operator and there is nothing to prove. If $b>a$, define

$$B := \frac{2}{b-a}\left[A - \frac{a+b}{2}I\right].$$

The spectrum of B is contained in the interval $[-1, 1]$. Using Lemma 2.1 we have the norm convergent Chebyshev expansion

$$A^{-1} = \frac{c_0}{2}I + \sum_{n=1}^{\infty} c_n T_n(B)$$

where c_n is given by (2.2). Now,

$$\begin{aligned}
\|A^{-1}\|_\infty &\leqslant \frac{1}{\sqrt{ab}} + \sum_{n=1}^{\infty} |c_n| \, \|T_n(B)\|_\infty \\
&\leqslant \frac{1}{\sqrt{ab}} + \sum_{n=1}^{\infty} \frac{2q^n}{\sqrt{ab}} (nm+1)^{1/2} \, \|T_n(B)\|_2 \qquad \text{(since } T_n(B) \text{ is } nm\text{-banded)} \\
&\leqslant \frac{2\sqrt{m}}{\sqrt{ab}} \left\{ 1 + 2 \sum_{n=1}^{\infty} q^n \sqrt{n} \right\} \qquad (\|B\|_2 = 1 \text{ forces } \|T_n(B)\|_2 = 1) \\
&\leqslant \frac{2\sqrt{m}}{\sqrt{ab}} \{1 + 4\Gamma(3/2)(2)^{3/2}(b/a)^{3/4}\} \qquad \text{(by Lemma 2.2)} \\
&\leqslant 33\sqrt{m}\, b^{1/4} a^{-5/4} = 33\sqrt{m}\, \|A\|_2^{1/4} \, \|A^{-1}\|_2^{5/4}. \quad \blacksquare
\end{aligned}$$

Note that the only use of the zero structure of A occurs in the second inequality; hence, the statement in the abstract.

We believe that the correct exponent on $\|A^{-1}\|_2$ is 1 but have not been able to prove it. If we somehow knew that the quantities $\|T_n(B)\|_\infty$ could be ignored, then the estimate $\sum_n |c_n| \leqslant 2/a$ would give such an exponent. Our computations with the bounds on the entries of A^{-1} given in [2] gave a bound on $\|A^{-1}\|_\infty$ of the form $\|A\|_2^{1/2} \, \|A^{-1}\|_2^{3/2}$ even in the tridiagonal case. If we assume that A is bi-infinite and Toeplitz, then we can translate Proposition 2.4 into a norm inequality for reciprocals of cosine polynomials. First, we introduce the norms.

For $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$ we define

$$\|f\|_\infty = \sup_{|z|=1} |f(z)|$$

and $\|f\|_A = \sum_n |c_n|$.

COROLLARY 2.5. *Let* $p(z) = \sum_{n=-m}^{m} a_n z^n$ *and assume that* $a_n = \bar{a}_{-n}$, $n \neq 0$, $a_0 \in \mathbb{R}$, *and* $0 < \alpha \leqslant p(z) \leqslant \beta$ *if* $|z| = 1$, *with the values* α, β *being taken on. Let* $r(z) = 1/p(z)$. *Then* $\|r\|_A \leqslant 33\sqrt{2m+1}\, \|r\|_\infty^{5/4} \, \|p\|_\infty^{1/4}$.

Proof. p is the symbol of a positive definite m-banded Hermitian matrix A: $A(i,j) = a_k$ if $|i-j| = k$. $\|p\|_\infty = \|A\|_2$, $\|p\|_A = \|A\|_\infty$, and r is the symbol of A^{-1} with like norm equalities. Now apply Proposition 2.4. $\blacksquare$

In principle one should be able to work out the details of the Toeplitz case in general. Several avenues of approach are possible. For example, a direct sharpening of the inequality $\|T_n(B)\|_\infty \leqslant \sqrt{nm+1}\, \|T_n(B)\|_2$ for Toeplitz B might be possible. Or one can try to bound the norm given by (2.3) directly. For the present we will give a bound for strictly diagonally dominant M-matrices. The proof is similar to that for totally positive Toeplitz matrices and as in that case does not assume bandedness, cf. [4].

PROPOSITION 2.6. *Let A be a bi-infinite Hermitian Toeplitz matrix such that $A(0,0) > \sum_{i \neq 0} |A(i,0)|$ and $A(i,0) \leqslant 0$ for all i. Then, $\|A^{-1}\|_2 = \|A^{-1}\|_\infty$.*

Proof. Let $p(z) = \sum_{i \neq 0} A(i,0) z^i$ be the symbol of A, with $r(z) = \sum_i b_i z^i$ the symbol of A^{-1}. The Neumann series representation of A^{-1} shows $b_i \geqslant 0$ for all i. Hence,

$$\|A^{-1}\|_2 = \|r\|_\infty = r(1) = \sum b_i = \sum |b_i| = \|A^{-1}\|_\infty \quad \blacksquare$$

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